TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
HOMEWORK 6

MATHIAS BRAUN AND WENHAO ZHAO

Homework 6.1 (Holomorphic interpolation*)**.** Let $(a_n)_{n\in\mathbf{N}}$ be sequence without an accumulation point and let $(b_n)_{n\in\mathbf{N}}$ be an arbitrary sequence (both in $\mathbf{C}$). Show there exists a holomorphic function $f\colon \mathbf{C} \to \mathbf{C}$ such that $f(a_n) = b_n$ for every $n \in \mathbf{N}$[1].

Homework 6.2 (Representation of meromorphic functions as quotients)**.** Let $h\colon \mathbf{C} \to \mathbf{C}$ be a meromorphic function, i.e. its singularities are isolated and only poles of finite order. Show that there exist two entire functions $f, g\colon \mathbf{C} \to \mathbf{C}$ with no common zeros with $h = f/g$.

Homework 6.3 (Weierstraß product theorem on open sets)**.** Let $U \subset \mathbf{C}$ be an open set and let $(a_n)_{n\in\mathbf{N}}$ be a discrete sequence in U. Set $o_n := \#\{k \in \mathbf{N} : a_k = a_n\}$ and assume $o_n < \infty$ for every $n \in \mathbf{N}$. We claim there exists a holomorphic function $f\colon U \to \mathbf{C}$ such that $Z(f) = \{a_n : n \in \mathbf{N}\}$ and $o_{a_n}(f) = o_n$ for all $n \in \mathbf{N}$. Moreover, as we shall see in the proof the function f can be taken as an infinite product. The argument splits into two steps. Similarly to the Mittag–Leffler theorem we recenter part of the Weierstraß factors and replace $E_n(z/a_n)$ by $E_n((a_n - c_n)/(z - c_n))$ for a suitable sequence $(c_n)_{n\in\mathbf{N}}$. We denote by $S' \subset \mathbf{C}$ the set of accumulation points of the sequence $(a_n)_{n\in\mathbf{N}}$. If $S' = \emptyset$ there is nothing to prove as we can apply Theorem 4.2 from the lecture notes. Hence assume that $S' \neq \emptyset$.

a. Suppose there exists a sequence $(c_n)_{n\in\mathbf{N}}$ in S' such that $|a_n - c_n| \to 0$ as $n \to \infty$. Show the infinite product
$$f(z) := \prod_{n=1}^{\infty} E_n\Big[\frac{a_n - c_n}{z - c_n}\Big]$$
converges locally normally on U and obeys $Z(f) = \{a_n : n \in \mathbf{N}\}$ and $o_{a_n}(f) = o_n$ for every $n \in \mathbf{N}$.

b. Split the set $S := \{a_n : n \in \mathbf{N}\}$ as in Lemma 2.7 from the lecture notes and use Lemma 2.8 to conclude the proof by combining a. and Theorem 4.2.

Homework 6.4 (First applications of Picard's little theorem)**.**

a. Show every meromorphic function f that omits three distinct values $a, b, c \in \mathbf{C}$ is constant.

b. Give an example of a meromorphic function that omits the two values $0, 1 \in \mathbf{C}$.

c. Let $f\colon \mathbf{C} \to \mathbf{C}$ be an entire function. Show $f \circ f$ has a fixed point unless f is affine but not linear, i.e. $f(z) = z + b$ for some $b \in \mathbf{C} \setminus \{0\}$[2].

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[1]**Hint.** Combine the Weierstraß product theorem with the Mittag–Leffler theorem for a suitable sequence of principal parts.

[2]**Hint.** Consider the assignment
$$g(z) := \frac{f(f(z)) - z}{f(z) - z}$$
Show it is constant and differentiate it. Then deduce $f' \circ f$ omits the value 0 and said constant. Finally, show this implies that f' is constant.